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MONTE CARLO AND NUMERICAL METHODS TO SOLVE THE $MA(q)$ TIME SERIES MODEL

Abstract. In this paper we will solve the nonlinear system of equations in the parameters of the $MA(q)$ time series model by Monte Carlo methods and by numerical methods.

When we identify the variance and the inter-covariances of time series, we obtain, dividing by variance, a quadratic nonlinear system of equation that does not contain the variance of wihite noise. We use only the autocorrelation function.

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Keywords: Moving average time series, Monte Carlo, nonlinear system of equations

1. Introduction

Before estimating coefficients of stationary time series model, we check first if the given time series is stationary. We use for it the Dickey-Fuller unit root test [2]. If the time series is not stationary, we stationarize it [1,5,7] by some methods, as differentiating method, moving average method, or by exponential smooth method.

After stationarization, we identify the coefficients of stationary time series, using the Yule-Walker algorithm for the $AR(p)$ time series,

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the innovations' algorithm for the $MA(q)$ time series and the Hannan-Rissanen algorithm for the $ARMA(p,q)$ time series [1,7].

Numerical methods to solve non-linear systems of equations are presented in [6]. Among these methods, we mention the contraction method and the Newton – Raphson method.

Solving a nonlinear system of equations and other mathematical problems by the Monte Carlo methods are presented in [8].

The idea of the innovations' algorithm is to consider first the time series to be white noise, next $MA(1)$, $MA(2)$, ..., and finally $MA(q)$. At each step we compute first the variance of the white noise, and the parameter θ_i decreasing on i from maximum possible value to one, as follows.

At the initial step the time series is considered white noise, with the variance of white noise equal to the variance of time series $\sigma_0^2 = \gamma_z(0)$.

At step m, when the time series is $MA(m)$, we compute

$$\begin{cases} \theta_{m-k,m} = \dfrac{\gamma_z(m-k) - \sum\limits_{j=0}^{k-1} \theta_{m-j,m} \cdot \theta_{k-j,k} \cdot \sigma_j^2}{\sigma_k^2} \\ \sigma_m^2 = \gamma_m(0) - \sum\limits_{0}^{m-1} \theta_{m-j,m}^2 \cdot \sigma_j^2 \end{cases}, \tag{1}$$

where $\gamma_z(0)$ is the variance of the time series, and $\gamma_z(j)$ is the autocovariance of order j. We notice that at step m the order of computation is $\theta_{m,m}$, $\theta_{m-1,m}$, ..., $\theta_{1,m}$, and finally σ_m^2.

The solution of the model is

$$X_t = a_t + \sum_{i=0}^{q} \theta_{i,q} \cdot a_{t-i}, \tag{2}$$

where a_t is a white noise with the variance σ_q^2.

The optimization problems on a given domain can be solved by the Monte Carlo methods if we know to simulate an uniform random variable on the given domain [8]. We simulate a big nuber of variables on the domain, and for each simulated value we compute the value of the function to be optimized. The solution is the generated value for which the function is minimum/ maximum. From here arises the ideea of solving nonlinear system of equations: we minimize the sum of squares of the differences between the left sides and right sides.

2. Methodology

As in the innovations' algorithm, we consider

$$X_t = a_t + \sum_{i=0}^{q} \theta_i \cdot a_{t-i} . \quad (3)$$

Denoting by γ_k the intercovariance of order k of the time series X_t (hence γ_0 is the variance) and by σ^2 the variance of a_t, we obtain

$$\begin{cases} \gamma_0 = \sigma^2 \left(1 + \sum_{i=1}^{q} \theta_i^2 \right) \\ \gamma_k = \sigma^2 \left(\theta_k + \sum_{i=1}^{q-k} \theta_i \cdot \theta_{k+i} \right) \end{cases} . \quad (4)$$

Dividing by γ_0 we obtain first

$$\frac{\theta_k + \sum_{i=1}^{q-k} \theta_i \cdot \theta_{k+i}}{1 + \sum_{i=1}^{q} \theta_i^2} = \rho_k , \ k = \overline{1,q} , \quad (5)$$

where ρ_k is the autocorrelation function. Finally, we obtain

$$\theta_k + \sum_{i=1}^{q-k} \theta_i \cdot \theta_{k+i} = \rho_k \left(1 + \sum_{i=1}^{q} \theta_i^2 \right), k = \overline{1,q}. \quad (5')$$

For the last equation in the above formula, when k=q the sum in the left side vanishes, and the equation becomes

$$\sum_{i=1}^{q-1} \theta_i^2 + \left(\theta_q - \frac{1}{2\rho_q} \right)^2 + 1 - \frac{1}{4\rho_q^2} = 0. \quad (6)$$

In order to have real solution for the last equation we must have $\rho_q < \frac{1}{2}$, which is a reasonable condition. We notice that in this case the last equation is the sphere with center $\left(0,...,0, \frac{1}{2\rho_q} \right)$ and radius $r = \sqrt{\frac{1}{4\rho_q^2} - 1}$.

For the Monte Carlo methods we simulate for q=1 a big number of values of θ, say 10000 in $(-1,1)$ and we choose θ such that we obtain the minimum of

$$\theta - \rho\left(\theta^2 + 1\right), \quad (7)$$

where $\rho = \rho_1$ and $\theta = \theta_1$. In fact, because θ and ϱ have the same sign, we simulate $\theta \in (0,1)$ for ϱ>0, respectively $\theta \in (-1,0)$ in the contrary case. In order to have a solution in $(-1,1)$ we must have first

$$\Delta = 1 - 4\rho^2 > 0, \quad (8)$$

hence $|\rho| < \frac{1}{2}$. The root with minus is in this case

$$\theta = \frac{1-\sqrt{\Delta}}{2\rho} = \frac{4\rho^2}{2\rho\left(1+\sqrt{\Delta}\right)} = \frac{2\rho}{1+\sqrt{\Delta}}. \quad (9)$$

We notice that, because the product of roots is, according Viète, one, the other root (with plus) is greater in absolute value than one.

If q>1, consider the parametrization

$$\begin{cases} \theta_1 = r \cdot \prod_{i=1}^{q-1} \cos t_i \\ \theta_k = r \cdot \sin t_{q-k+1} \cdot \prod_{i=1}^{q-k} \cos t_i , 1 < k < q , \\ \theta_q = \frac{1}{2\rho_q} + r \cdot \sin t_1 \end{cases} \tag{10}$$

where for $1 \le i < q$ we have $t_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and $t_{q-1} \in [0, 2\pi]$. Therefore we simulate sets $(t_1, ..., t_{q-1})$ uniform in the corresponding intervals, and next we compute θ_k and the differences between left and right sides for the other equations. This is the parametrisation of the sphere of dimension *q*. For q=3 we have the well known sphere. In this case we simulate for each point a pair (t_1, t_2). We notice that, for having at least one index such that $1 \le i < q$ as above, we must have q>2. In the case q=2 we obtain the circle with the center and radius computed above for (6). For q=1 we simulate 10000 sets of values $(t_1, ..., t_{q-1})$ - 10000 numbers in $[0, 2\pi]$ in the case of circle (q=2). For each simulated set, we compute according the above parametrization the carthesian coordinates $(\theta_1, ..., \theta_q)$. We compute for each such point on the sphere the sum of squares of differences between left sides and right sides in (5'), except the last equation, used for simulation. We choose the parameters $(\theta_1, ..., \theta_q)$ such that the mentioned sum of squares is minimum. The sum of squares, *S*, is a random variable having the cumulative distribution function *F*. The error for a minimization problem in the general case, and for the minimum sum of squares in our case is given by

$$\varepsilon = F^n(S). \tag{11}$$

Next proposition presents the cumulative distribution function *F* from (11), for q=1.

Proposition 1*.*

Suppose that $|\rho| < \frac{1}{2}$ *in order to have a solution* $\theta \in (0,1)$ *in (7).*

For q=1*, we simulate the values* $\theta \in (0,1)$ *if* $\rho = \rho_1 > 0$ *and* $\theta \in (-1,0)$ *in the contrary case. We compute the difference* $dif = \theta - \rho(\theta^2 + 1)$*. The cumulative distribution function in α>0 is*

$$F(\alpha) = (\theta_2 - \theta_1)^n,$$

where n *is the number of simulated* Θ*, and* Θi *are such that* $dif = \pm\sqrt{\alpha}$.

For numerical methods, we consider $\theta^{(0)} = 0.5$ as initial solution. We solve the equation

$$\rho\theta^2 - \theta + \rho = 0 \tag{12}$$

by the tangent method (Newton Method). Of course, this equation can be solved analyticaly, obtaining the solution

$$\theta_{12} = \frac{1 \pm \sqrt{1 - 4\rho^2}}{2\rho}. \tag{13}$$

Because, according the relations of Viète, we have the product of roots equal to one, we choose the root in absolute value less than one. We notice that from $\Delta = 1 - 4\rho^2 > 0$ we obtain also $\rho_1 = \rho$ less in absolute value than 0.5 as in the general case.

For the contraction method, we compute successively

$$\theta^{(k+1)} = \rho\left(\left(\theta^{(k)}\right)^2 + 1\right). \tag{12'}$$

In the case q>1 we consider for the initial solution $\theta_1^{(0)}$ as the solution of (13) , and $\theta_i^{(0)} = 0$ for the indexes $i = \overline{2,q}$.

For the Newton-Raphson method, we obtain the Jacobean matrix *J* and the right sides *fx*. For the matrix *J* we have

$$J_{ii} = \begin{cases} 1 - 2\rho_i\theta_i \; if \; 2i > q \\ 1 - 2\rho_i\theta_i + \theta_{2i} \; otherwise \end{cases}, \; and \tag{13}$$

$$J_{ij} = \begin{cases} -2\rho_i\theta_j \; if \; q - i < j \le i \\ \theta_{j+i} - 2\rho_i\theta_j \; if \; j \le \min(i, q-i) \\ \theta_{j-i} + \theta_{i+j} - 2\rho_i\theta_j \; if \; i < j \le q - i \\ \theta_{j-i} - 2\rho_i\theta_j \; otherwise \end{cases}. \tag{13'}$$

For the values of *fx* we have, according (4′)

$$f_i = \theta_i - \rho_i\left(1 + \sum_{j=1}^{q}\theta_i^2\right) + \sum_{j=1}^{q-i}\theta_j\theta_{j+i} \,. \tag{13''}$$

Proposition 2.

The nonlinear system (4′) can be solved by the Newton-Raphson method with any initial solution interior to the spheres having the centers and radius given above, for $q = \overline{2, q_{\max}}$, with zeroes for components $\theta_i,\ i > q$.

For the contraction method we consider $\theta^{(0)} = \left(\theta_1^{(0)}, \ldots, \theta_q^{(0)}\right)$ in the interior of the sphere used for simulation, and at each step

$$\begin{cases} \theta_k^{(n+1)} = \rho_k\left(1 + \sum_{i=1}^{q}\left(\theta_i^{(n)}\right)^2\right) - \sum_{i=1}^{q-k}\theta_i^{(n)} \cdot \theta_{k+i}^{(n)}, \; for \, 1 \le k < q \\ \theta_q^{(n+1)} = \rho_q\left(1 + \sum_{i=1}^{q}\left(\theta_i^{(n)}\right)^2\right) \end{cases}. \tag{14}$$

Proposition 3.
Denote by $\rho=\max_{1\le i\le q}\left|\rho_i\right|$. *If* $\rho<\frac{1}{2}$, *the nonlinear system* (5′) *can be solved by the contraction method with any initial solution interior to the spheres having the centers and radius given above, for* $q=\overline{2,q_{\max}}$, *with zeroes for components* $\theta_i,\ i>q$.

The formula (14) is the Jacobi contraction. The Gauss-Seidel contraction is

$$\begin{cases}\theta_1^{(n+1)}=\rho_1\left(1+\sum_{i=1}^{q}\left(\theta_i^{(n)}\right)^2\right)-\sum_{i=1}^{q-1}\theta_i^{(n)}\cdot\theta_{1+i}^{(n)}\\ \theta_k^{(n+1)}=\rho_k\left(1+\sum_{i=1}^{k-1}\left(\theta_i^{(n+1)}\right)^2\right)+\rho_k\left(\sum_{i=k}^{q}\left(\theta_i^{(n)}\right)^2\right)\\ -\sum_{i=1}^{q-k}\theta_i^{(m(i,k))}\cdot\theta_{k+i}^{(n)},\ for\ 1<k<q\\ \theta_q^{(n+1)}=\rho_q\left(1+\sum_{i=1}^{q-1}\left(\theta_i^{(n+1)}\right)^2\right)+\rho_q\left(\theta_q^{(n)}\right)^2\end{cases},\ where \tag{14′}$$

$$m(j,k)=\begin{cases}n+1\ if\ \ j<k\\ n\ if\ \ j\ge k\end{cases}. \tag{14″}$$

For the numerical methods used in this paper, namely the Newton-Raphson method and the contraction method, we have to start with an initial solution θ_0. We consider two approaches. First one (the above presented approach) consists in using an initial solution θ_0 for given *q* without tacking into account the solutions for less values of *q*. The other approach uses for $q>2$ the initial solution

$$\left(\theta_{1,q-1},\ldots,\theta_{q-1,q-1},0\right),\ where \tag{15}$$

$$\left(\theta_{1,q-1},\ldots,\theta_{q-1,q-1}\right) \tag{15′}$$

is the solution obtained for $q-1$.

As error, for the above methods we consider in the Newton-Raphson case the absolute value of $f(x)$, since we know that, theoretically, $f(x)=0$. In the contraction method case we consider as error the Euclidean distance between the last two estimations of the vector θ.

After we have estimated by any method the coefficients θ_i for a given $\mathrm{MA}(q)$ time series, we estimate the variance, according (4): we divide the variance of time series, γ_0, by the sum between parentheses.

3. Application

Example 1.

Consider the Consumer Price Index (CPI) in the period January 2010 – December 2018, monthly data (108 values) in Romania [ipc].

If we apply the Dickey-Fuller test [2] the value of Φ varies from -0.045016 (non significant) for the model 3, -0.030751 for the model 2, and -0.00212 for the model 1 (last two vales significant). For the first difference, the maximum Student statistics is -7.356323 for the model 1, which is significant 1%. Therefore, the time series is $I(1)$.

In the case q=1 we obtain by the analytical method (second degree equation) $\theta_1 = -0.2340310762$ and $\sigma_a^2 = 0.3243509072$. By the Monte Carlo method, after 10000 simulations, we obtain $\theta_1 = -0.2340464492$ and $\sigma_a^2 = 0.3243486945$.

For numerical methods we consider the threshold of the error $\varepsilon = 10^{-8}$, because a real number in simple precision on computer has eight digits. By the tangent method, we obtain after 3 iterations $\theta = -0.2340310759$ and $\sigma_a^2 = 0.3243509072$. By the contraction method, we obtain after 9 iterations $\theta = -0.2330310768$ and $\sigma_a^2 = 0.3243509071$.

In Table 1 we present for the first difference the $\mathrm{MA}(q)$ coefficients obtained using the innovations' algoritm, the Monte Carlo methods (10000 simulations), and the contraction method, for $q = \overline{1,5}$. For the numerical methods we consider the threshold for the errors $\varepsilon = 10^{-8}$. We mention on the last row for each numerical method and q the number of iterations. We consider in this case the initial solution for numericai methods the first approch: we do not take into account the solutions for previous values of q.

For the second approach, the results are presented in Table 2. The number of iterations for each $q>2$ means the number of iterations starting with the final solution for $q-1$ as initial solution.

Table 1

***MA(q)* coefficients for ΔXt, $q=\overline{2,5}$**

q	Innovations' algorithm	Newton-Raphson	Contractions' method		Monte Carlo methods
			Jacobi	Gauss-Seidel	
2	$\begin{pmatrix}-0.19098\\-0.18161\end{pmatrix}$	$\begin{pmatrix}-0.2000851363\\-0.1958517615\\3\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.2000851408\\-0.1958517628\\6\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.2000851408\\-0.1958517628\\6\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.2003501803\\-0.1958725144\end{pmatrix}$
3	$\begin{pmatrix}-0.19039\\-0.17479\\-0.06956\end{pmatrix}$	$\begin{pmatrix}-0.1904482236\\-0.1809261397\\-0.0747472561\\4\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1904482276\\-0.1809261397\\-0.0747472559\\9\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1904482282\\-0.1809261398\\-0.0747472556\\9\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1924663623\\-0.1817797451\\-0.0748233366\end{pmatrix}$
4	$\begin{pmatrix}-0.19031\\-0.17199\\-0.05979\\-0.05728\end{pmatrix}$	$\begin{pmatrix}-0.190657763\\-0.1724122593\\-0.0629667196\\-0.0615089327\\4\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1906577621\\-0.1724122594\\-0.0629667196\\-0.0615089329\\9\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1906577623\\-0.1724122593\\-0.0629667196\\-0.0615089327\\10\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.19031\\-0.17199\\-0.05979\\-0.05728\end{pmatrix}$
5	$\begin{pmatrix}-0.19021\\-0.17195\\-0.05848\\-0.05339\\-0.0294\end{pmatrix}$	$\begin{pmatrix}-0.1902561798\\-0.1723723119\\-0.0586709948\\-0.0554825038\\-0.0315553431\\5\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1902561799\\-0.1723723118\\-0.0586709948\\-0.0554825038\\-0.0315553431\\10\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1902561792\\-0.1723723119\\-0.0586709948\\-0.0554825038\\-0.0315553431\\10\,iterations\end{pmatrix}$	$\begin{pmatrix}-0.1981159039\\-0.1703646239\\-0.0757286328\\-0.0436678251\\-0.0305838236\end{pmatrix}$

Table 2

The results of numerical methods for *MA(q)* coefficients if we use the previous solutions as initial ones

q	Newton-Raphson method	Contractions' method	
		Jacobi	Gauss-Seidel
2	$\begin{pmatrix} -0.2000851363 \\ -0.195851765 \\ 3\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.2000851408 \\ -0.1958517628 \\ 6\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.2000851408 \\ -0.1958517628 \\ 6\ iterations \end{pmatrix}$
3	$\begin{pmatrix} -0.1904482293 \\ -0.1809261398 \\ -0.074747256 \\ 5\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.190448226 \\ -0.1809261399 \\ -0.0747472562 \\ 8\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.1904482278 \\ -0.1809261398 \\ -0.074747256 \\ 9\ iterations \end{pmatrix}$
4	$\begin{pmatrix} -0.1906577642 \\ -0.1724122593 \\ -0.0629667195 \\ -0.0615089328 \\ 5\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.1906577622 \\ -0.1724122594 \\ -0.0629667196 \\ -0.0615089328 \\ 9\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.1906577629 \\ -0.1724122593 \\ -0.0629667196 \\ -0.0615089327 \\ 9\ iterations \end{pmatrix}$
5	$\begin{pmatrix} -0.1902561794 \\ -0.1723723119 \\ -0.0586709948 \\ -0.0554825038 \\ -0.0315553431 \\ 5\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.1902561796 \\ -0.1723723119 \\ -0.0586709948 \\ -0.0554825038 \\ -0.0315553431 \\ 10\ iterations \end{pmatrix}$	$\begin{pmatrix} -0.1902561801 \\ -0.1723723119 \\ -0.0586709948 \\ -0.0554825038 \\ -0.0315553431 \\ 9\ iterations \end{pmatrix}$

The variances of the white noise is for the above methods are presented in Table 3. For the numerical methods if we take into account the previous solutions (corresponding to Table 2) ar identical in the Newton-Raphson case, the last digit 3 becomes 5 for q=3 and the last digits 79 become 80 for q=5 in the Jacobi contraction case, and last digit 2 becomes 3 for q=3 and the last digits 80 become 79 for q=5 in the Gauss-Seidel contraction case.

We notice that for the contraction method we can have more iterations for small q in the Jacobi case. But for higher values of q the Gauss-Seidel contraction makes an serious improvement to the Jacobi contraction. In Table 4 we present the numbers of iterations if we use/ we do not use the previous values, for $q_{\max} = 15$ and all three numeric methods. In this table, Yes means we take into account the previous results (for previous values of *q*), and No means we do not.

In Table 5 we present the last errors in the cases of the two numerical methods and the two cases (using/ no using the previous values). *nrit* represents the no. of iterations such that the error becomes less than 10^{-8}.

We mention that in the case q=1 we obtain the previous error $3.257 \cdot 10^{-5}$ with the tangent method and $5.383 \cdot 10^{-8}$ with the contraction method. The last errors are $2.931 \cdot 10^{-10}$, respectively $5.591 \cdot 10^{-9}$.

Table 3

The variance of the white noise for $q=\overline{1,5}$ in the cases of inovations' algorithm, and our three numerical methods and our Monte Carlo methods

Method	**q=1**	**q=2**	**q=3**	**q=4**	**q=5**
Innovations' algorithm	0.3252733751	0.315249914	0.3143827599	0.3137850941	0.3138204017
Newton-Raphson	0.3243509072	0.3172461365	0.3183680058	0.3185955435	0.3187138367
Jacobi contractions'	0.3243509071	0.3172462245	0.3183680873	0.3185956129	0.3187139079
Gauss-Seidel contractions	0.3243509071	0.3172462245	0.3183680872	0.3185956128	0.318713908
Monte Carlo	0.3243486945	0.3172126144	0.3180443697	0.315591096	0.3177001777

Table 4

The number of iterations for $q_{\max} = 15$ and $q=\overline{6,15}$ if we take/ we do not take into account previous solutions

Method \ q	Yes/ no	6	7	8	9	10	11	12	13	14	15
Newton-Raphson	Yes	6	7	6	7	7	8	8	8	9	10
	No	5	6	6	5	6	6	6	6	6	8
Jacobi contractions'	Yes	14	20	19	21	26	27	34	36	33	49
	No	14	21	22	24	28	31	37	41	40	54
Gauss-Seidel contractions'	Yes	10	12	12	12	14	14	15	15	15	20
	No	10	12	12	13	15	15	16	17	16	21

Table 5

The last errors in the case of using/ not using the previous solutions for the numerical methods, $q=\overline{2,5}$

Method	Yes/ no	Nrit-1/ nrit	q=2	q=3	q=4	q=5
Newton-Raphson	No	nrit-1	$8.106*10^{-5}$	$1.513*10^{-6}$	$9.793*10^{-8}$	$4.787*10^{-7}$
		nrit	$4.789*10^{-9}$	$6.738*10^{-9}$	$5.923*10^{-10}$	$4.453*10^{-10}$
	Yes	nrit-1	$8.106*10^{-5}$	$4.75*10^{-5}$	$3.529*10^{-7}$	$1.236*10^{-7}$
		nrit	$4.789*10^{-9}$	$1.991*10^{-9}$	$1.935*10^{-9}$	$5.729*10^{-13}$
Jacobi contractions'	No	nrit-1	$7.138*10^{-7}$	$1.764*10^{-8}$	$1.11*10^{-7}$	$4.211*10^{-8}$
		nrit	$3.369*10^{-9}$	$3.004*10^{-9}$	$9.542*10^{-9}$	$5.659*10^{-9}$
	Yes	nrit-1	$7.138*10^{-7}$	$1.228*10^{-7}$	$5.358*10^{-8}$	$1.198*10^{-8}$
		nrit	$3.369*10^{-9}$	$6.634*10^{-9}$	$5.08*10^{-9}$	$1.902*10^{-9}$
Gauss-Seidel contractions'	No	nrit-1	$7.138*10^{-7}$	$7.494*10^{-8}$	$1.175*10^{-8}$	$2.069*10^{-8}$
		nrit	$3.369*10^{-9}$	$7.94*10^{-9}$	$1.152*10^{-9}$	$2.072*10^{-9}$
	Yes	nrit-1	$7.138*10^{-7}$	$3.436*10^{-8}$	$5.775*10^{-8}$	$7.736*10^{-8}$
		nrit	$3.369*10^{-9}$	$3.817*10^{-9}$	$5.66*10^{-9}$	$7.761*10^{-9}$

4. Conclusions

In [3] a long-term time series model for backbone trafic is presented. The used model is $SARIMA(p,d,q)\times(P,D,Q)_s$. Portmanteau tests for residuals are used to choose between ARMA models.

In [4] there are presented Bayesian simulation techniques for time series, namely the Gibbs algorithm. The simple vs. multi-state sampling are compared in the mentioned article.

In our paper, opposite the innovations' algorithm, where we estimate at each iteration *k* first θ_k, and next $\theta_{k-1},\ldots, \theta_1$ (and finally the variance), in the presented two contraction methods we estimate first θ_1, and next $\theta_2,\ldots, \theta_k$. An improvement is for all three methods the elimination of the variance, according (5′). Only after we have estimated the values $\theta_1,\ldots, \theta_q$ we estimate the variance using (4).

Because the error is 10^{-8}, the first seven digits for the estimated values of θ_i are the same in each case of q_{max} and q for all three numerical methods. Comparing the two approaches, we have differences only for $q>2$. For the estimated values of θ, we have the maximum difference in absolute value $4.3\cdot10^{-9}$ for $q=\overline{3,5}$ and $8.4\cdot10^{-9}$ for $q=\overline{6,15}$ in the case of Newton-Raphson method, $1.6\cdot10^{-9}$ for $q=\overline{3,5}$ and $3.3\cdot10^{-9}$ for $q=\overline{6,15}$ in the case of Jacobi contraction method, and $9\cdot10^{-10}$ for $q=\overline{3,5}$ and $2.4\cdot10^{-9}$ for $q=\overline{6,15}$ in the case of Gauss-Seidel contraction method. The absolute values of differences for the variances of white noise are for $q=\overline{3,5}$ in the three cases $5\cdot10^{-10}$, $2\cdot10^{-10}$, respectively 10^{-10}. Similarly, we obtain for $q=\overline{6,15}$ the maximum differences in absolute values $9\cdot10^{-10}$, $9\cdot10^{-10}$, respectively $5\cdot10^{-10}$.

Comparing the two contraction methods we obtain the above differences in absolute values for θ being $2.2\cdot10^{-9}$ for $q=\overline{3,5}$ and $5\cdot10^{-9}$ for $q=\overline{6,15}$. For the variance of the white noise, the absolute values of differences are $3\cdot10^{-10}$ for $q=\overline{3,5}$ and $1.1\cdot10^{-9}$ for $q=\overline{6,15}$.

Comparing all three numeric methods we obtain the above differences in absolute values for θ being $7.6\cdot10^{-9}$ for $q=\overline{3,5}$ and $1.07\cdot10^{-8}$ for $q=\overline{6,15}$. For the variance of the white noise, the absolute values of differences are $9\cdot10^{-10}$ for $q=\overline{3,5}$ and $1.5\cdot10^{-9}$ for $q=\overline{6,15}$.

The number of iterations in the contraction case is, as we see in Table 4, greater if we do not take into account the previous results. In the case of Jacobi contraction, we have three additional iterations if $q\in\{8,9,12\}$, four additional iterations if $q=11$, five additional iterations for $q=13$ or $q=15$, and even seven additional iterations if $q=14$.

Between the two contraction methods, we have generally the lowest values for the number of iterations in the Gauss-Seidel case, as expected. This because the estimated values of θ are used immediately in this case, while in the Jacobi case we use the estimated value θ_i with $i<q$ only after we estimate θ_q at a given iteration. Other case with low numbers of iterations, lower even the Gauss-Seidel case is the case of Newton-Raphson method. In this case the explanation comes from the other definition of error: the absolute value of $f(\theta)$, which is decreased by the factors $\rho_i<1$.

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