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RUSSELL'S FORMAL LOGIC

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Abstract. Russell's formal logic has been criticized by some formal logicians as not having fully precise syntax (form). However, he is guided by the principle of formal logic, (Form) that proof depends entirely on form, independently of content. This is a profoundly questionable view of proof. It conflicts with recognizing that (i) "The set of dogs is a non-self-member" and (ii) "The set of cats is a non-self-member" are attributing different properties in spite of having the same predicate. The predicate form Rx: "x is a non-self-member" is common to (i) and (ii) but the content of the predicate, the property attributed, is different. This is the key to resolving Russell's paradox and is unavailable to Russell due to his commitment to (Form). His "Vicious Circle Principle" is not well stated and its concern with content conflicts with (Form). But a proper understanding of the principle does provide an answer to the "Truthteller Paradox". That allows me to close on a note of agreement with Russell.

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1. Russell's *formal* logic reflects the defining idea of traditional formal logic – Whether a proof is good, valid, is determined by its logical form alone, independently of its content. (However, his logic did not always conform to this. Goedel criticized the syntax of PM as "a decided step backward from Frege" and Quine complained of PM confusing predicates with properties.)

2. "Proof" in a common "informal" use, marks right (as opposed to mere) persuasion, which is relative to an audience. A proof for audience A

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may not qualify for audience B because B cannot fairly be expected to understand the whole of premises, rules and steps to the conclusion. If an understanding audience accepts the premises and the rules of inference and is able to effectively apply those rules, then it counts against their reasoning well if they do not accept the conclusion. This bad mark can be answered in some cases. Reaching or nearing the conclusion may shock the audience into reviewing its acceptances. Moore famously held that the primary evaluative term "good" is not definable. Unfortunately, his reasoning applies equally to "definable". It also applies to informal "proof". This is not to say these things are indefinable, but to excuse not attempting to define.

3. The primary problem about the concept of good argument from premises to a conclusion is explaining how this relation can hold between falsehoods and independently of the beliefs (or supposings) of the audience.

4. The idea of formal logic that proof, as well as valid argument, is determined by logical form independently of content, leads to an extraordinary finessing of this problem. The associated idea of formal logical validity, unlike the fully evaluative meaning, can be checked mechanically. If it did capture the evaluative meaning, this would be a triumph of naturalism. In dispensing with content in argument assessment, it is also a boost to nominalism.

5. In formal systems in general, proof is defined in the syntax. (As observed above, the syntax of PM has been criticized as not perfectly satisfactory in this respect.) A proof from premises is a list of wfs such that each entry is either an axiom, or a premise or follows from earlier entries by a rule of inference (which can be checked mechanically). What is it to be an axiom, how do axioms differ from rules of inference, and what is it to be a premise? Russell was certainly not averse to discussing such vague questions. But they are made irrelevant in the formal definition of proof. An axiom is from a recursively enumerable list called "axioms" and a premise is a wf which is not an axiom. An n-place rule of inference is a recursive set R(x,y) where x is an n-tuple of wfs and y is a wf and y is said to "follow from" x. PM did not clearly realize this ideal. This simplification of the idea of proof in a formal system invites relativising provability to formal systems. If a given formal system is thought to capture all correct reasoning, then its advocates may conclude they have gotten beyond the relative. Russell may once have had that attitude towards PM. Nowadays there are advocates of "Logical Pluralism".

6. Russell's attitude toward "classical logic" is complex. He says "Take for example, the law of excluded middle, in the form 'all propositions are true or false'. If from this law we argue that because the law of excluded middle is a proposition, therefore, the law of excluded middle is true or false, we incur a vicious circle fallacy. 'All propositions' must be in some way limited before it becomes a legitimate totality, and any limitation which makes it legitimate must make any statement about the totality fall outside the totality." (PM Intro. Chapter II, section I)

7. That is badly worded. The phrase "all propositions" is as legitimate a totality as any phrase. Russell is thinking of totalities of propositions, not totalities of letters or words. Calling a totality illegitimate is a loose way of saying there is no such totality. The argument (TF) (i) "All propositions are true or false; (ii) that is a proposition, therefore, (iii) it is true or false" is one argument. I take it as a necessary condition of circular reasoning that there be at least two arguments. It might be argued that TF begs the question, which is often conflated with circularity, and is fallacious. The charge of question begging calls for identifying the question. For TF, the plausible candidate is the question (Q) whether premise (i), the LEM, has one of the values, true or false. (This should not be confused with the question whether the LEM is true.)

The closest literal reading I can make of Russell's remarks is as misrepresenting his intended message. On this reading, if we have identified a proposition P and are arguing with someone who holds that P is neither true nor false, it would beg the question to appeal to the LEM. Granting that, it might then be held that the LEM is pointless, since it will never help settle an issue over whether a given proposition has a truth value. Whatever we say about that does not support the idea that there is no totality of all propositions. I don't think Russell thought it did either – he was misspeaking. He meant to convey that there cannot be a totality of all propositions which includes a proposition about all propositions.

Why not? It would somehow involve objectionable circularity. That TF begs the question is far from clarifying, let alone justifying, that complaint.

8. PM refers to theorem *2.11: ⊢. pv~p, as "The law of excluded middle". 'p' is a variable which can be replaced by any wf of PM. That, of course, is not part of the "content" of *2.11. It is part of the content of PM. As such it would not be read as (i) "all propositions are either true or false", but as (ii) "For every p, either p or not-p". "True" and "false" are not allowed in PM without subscripts. Negation is not restricted in that way. But (ii) uses 'p' as a sortal variable, restricted to propositions (or rather, to wfs). To get back to uniform quantifiers requires such as (iii) "For every x, if x is a proposition, then either x is true or x is false". Without "true" and "false" most substitutions would be nonsensical. If it is held that every proposition, that is, premise or conclusion, can be "represented" in PM as far as its logical powers are concerned, by some wf, then that claim about *2.11 is a proposition about every proposition. (Note that, if challenged to defend the claim that (pv~p) v ~(pv~p), appeal to 2.11 could be fairly accused of begging the question.)

9. PM was inspired by Russell's discovery of an inconsistency in the assumption that every predicate Fx determines an extension. If r is the extension of the predicate ~(xϵx), then (rϵr)↔~(rϵr), a contradiction. His solution was that there is no such predicate in PM, a model of a logically perfect language, and languages with such a predicate (English has Rx="x is a set which is not a member of itself") are defective. Why? One line is that "x belongs to y" just means "x belongs to the one level higher set y" so that "x belongs to the one level higher set, itself" is nonsense. Why must it "mean" that and why is that "meaning" nonsense? Why not just false, so ~(xϵx) is trivially true and determines the universal set? That would involve trouble (Cantor's Paradox). That a saying involves trouble is a weak basis for denying it can be said.

10. A similar line is the argument that, since taking R(r) as a premise leads to a contradiction, it follows that R(r) is nonsense and cannot be a premise. That seems to entail that since taking (Root) "The square root of 2 is equal to a quotient of integers in lowest terms" as a premise leads to a contradiction, it follows that (Root) is nonsense and cannot be a premise. That is a paradoxical verdict.

11. First order set theories such as ZF and NBG have the predicate ~(xϵx) but pretend it does not determine a set. (NBG has it determine a "proper class".) In any interpretation I of FOL, a wf with one free variable, such as ~(xϵx), determines a subset of the domain of I. This set is just not allowed to be an element of the domain. Efficient, but philosophically crude.

12. The right answer is that the predicate 'Rx' in application to sets does not express a common property. Applied to the set of dogs it expresses not being a dog (and having no other property that determines that set), to the set of cats, not being a cat, etc. Unapplied, it does not express any property and for that reason, does not determine a set, so Rr has a non-referring subject term. Unfortunately, this point is not widely recognized. (This answer to Russell's Paradox also applies to Cantor's. Cantor's argument for the thesis (as distinguished from the well-established theorem in first order set theories) that the power set of a set is always larger depends on a predicate essentially similar to the Russell predicate.)

13. So we have been left with Russell's Vicious Circle Principle (VCP), an attempted formulation of which is described (decried?) above (7). After dismissing the law of excluded middle, he continues:

> "The principle which enables us to avoid illegitimate totalities may be stated as follows: 'Whatever involves *all* of a collection must not be one of the collection'; or, conversely: 'If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.' We shall call this the 'vicious-circle principle,' because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities." (1910, 2nd ed chapter II section I)

14. That implies there is a certain collection (that is, a set – Russell also speaks in terms of a set "having no total") which "has no total". That is nonsense. A collection, set, just is a total(ity). Defining, say, the tallest man in a regiment requires reference to a total with a "member only definable in terms of that total". (You may disagree if you think of some actual regiment and walking up to the tallest man and identifying him. That is not defining the role of tallest man in a regiment.) It involves restrictions on defining "totalities" which had to be relieved with the Axiom of Reducibility.

There is no set of non-self-membered sets because the predicate 'Rx' does not express a property common to the (many) sets of which it can be predicated truly. (For any predicate F, there is such a property as being a thing to which the predicate F applies truly, but this is not a logically significant commonality, due to ambiguity and is not expressed by the predicate F anyway.)

15. The ideal of formal logic, that validity is independent of content, conflicts with recognizing this. "Rx" may be a form common to many predications, but in a given interpretation of a formal system, "Ra" cannot be taken to attribute to the denotation of 'a' something different from what it attributes to 'b' with "Rb". That point requires attention to content.

16. This defect of formal responses to Russell's Paradox is even clearer with the Liar. Eubulides says "What I am saying is false". (Make this "What Eubulides is saying is false", assuming we all know he refers to himself in that way. And waive all assumptions about the historical character of that name.) Noting he is not talking about some other saying, he is attributing being false to what he is saying – his content. It is summarized as that what E. is saying is false. So, insofar as he is saying anything, he is saying that what E. is saying is false and that it is false that what E. is saying is false, thus contradicting himself. Name what he says 'a' and in saying the words "a is false" he thereby achieves this self-contradiction. For that reason, I say that a is false, uttering the same words. (It can be the same token, written on a placard.) The formal approach requires that my assertion of those words must get the same treatment as Eubulides' assertion of them, since the reference and the predicate are the same. But my use is not to assert what Eubulides claimed to assert. I am referring to the words and to the series of expansions that result in trying to identify their content and to the fact that this series is inconsistent. Unlike Eubulides, I am not asserting that content. Attention to form alone cannot draw this essential distinction. For a wf, who asserts it or supposes it, the user, is out of the picture.

17. These considerations apply to the many variations on the Liar – two and more step versions, alethic modal versions, infinite series versions, etc. The applications vary from case to case. Epistemic and doxastic versions such as "No one knows this sentence is true" or "No one believes this sentence is true" require somewhat different treatment. They all require attention to the relation of the user to what is said. To apply to such as "A. The sentence A is not true" or "'Yields a falsehood when appended to its own quotation' yields a falsehood when appended to its own quotation", where there appears to be no user, calls for treating sentences as the sayers, and this needs elaboration which cannot be presented here.

18. Identifying the content of Eubulides' saying leads to regress of a sort that should be the intended target of the Vicious Circle Principle. "What I am saying is false." "What is the saying?" "That what I am saying is false." "And what is that saying?" There is a series that circles back to the original words. This is not circular argument. It is circular explanation of content. No saying that does not refer to a saying is ever reached. But the inconsistency of the series is adequate reason to pronounce the saying false.

19. For the Law of Excluded Middle it is possible to generate a similar regress in reply to request for instances of "all propositions", but if we understood the law, we might see overriding reason to judge it true. (Unfortunately, we cannot presume we have such a shared understanding. In fact, I presume we do not.) If Eubulides had said "What I am saying is true", that attempt at saying would be defenseless against a properly understood version of the VCP. There is no possible basis for assigning a truth value. Not having a truth value is not a truth value. (There is the point that there is no inconsistency. That is a virtue shared by "Blah-blah-blah".) The circularity in attempting to explain the content is not overridden by any feature. It is nice to close on this note of agreement with Russell.

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